Long-Wave Optics

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Abstract— **In this paper we set out the bases of the near-complete analytical methodology that now exists for the design of long-wave optical systems. We follow the Gaussian beam-mode treatment of free-space propagation, extend it to cover the transformations produced by conic-section reflectors or lenses, and incorporate both the propagation steps and the lens transformations into a matrix formulation readily applicable to networks of such reflectors or lenses. We demonstrate in the process the theorems of Fourier Optics and keep explicit the vectorial properties of the beam-fields. We show how recent formulations of partial coherence have made it possible to include partially-coherent beams in the same methodology. For the design of high-performance systems, the inclusion of higher-order mode-dispersion must be fully understood, the vector properties must be recoverable, and the paraxiality on which the methodology rests must be critically assessed. This paper gives emphasis to these aspects and presents a single systematic formulation embracing all the elements.**

I. Introduction

Major applications of electromagnetic waves in the 100–1000 GHz range are developing, in earth-remote-sensing, high-temperature plasma heating and diagnostics, high-definition radar, and astronomy. These applications require high performance levels not only in the receivers and sources but also in the directional quality of the beams that are transmitted or received. These are formed by optical systems which, in addition to shaping the beams, may be required to provide various signal conditioning or analysis steps, such as frequency and polarization multiplexing, radiometric calibration or spectral analysis.

Long waves spell diffraction, of course. Evaluating classical diffraction integrals for a beam passing through a succession of conic-section reflectors or lenses would be a laborious way to approach design and optimization. It as well that there is now, in our view, a near-complete analytical methodology for the design of long-wave optical systems. This paper will set out the bases of it. It follows the Gaussian beam-mode treatment of free-space propagation, extends it to cover the transformations produced by conic-section reflectors or lenses, and incorporates both the propagation steps and the lens transformations into a transfer matrix formulation readily applicable to networks of such reflectors or lenses. It demonstrates in the process the theorems of Fourier Optics and it keeps explicit the vectorial properties of the beam-fields. Moreover, it is not restricted to coherent beams; recent formulations of partial coherence have made it possible to include partially-coherent beams in the same methodology.

The various elements in this Long-wave Optics have been separately developed over the last twenty years or so. Some special-case aspects have become familiar—in particular the analysis of the propagation of a scalar pure-Gaussian beam through a train of lenses. But for the design of high-performance systems, the inclusion of higher-order mode-dispersion must be fully understood, the vector properties must be recoverable, and the paraxiality on which the methodology rests must be critically assessed. This paper gives emphasis to these aspects in presenting a single systematic formulation of Long-wave Optics embracing all the elements.

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II. Beam-Mode Optics

2.1 Modal Propagation

The propagation in free-space of a coherent field, of angular frequency ω, is governed by the Helmholtz wave-equation

$$\nabla^2 \boldsymbol{F} + k^2 \boldsymbol{F} = 0 \tag{1}$$

where $\boldsymbol{F}$ is any one of the electromagnetic fields $\boldsymbol{E}$, $\boldsymbol{H}$, $\boldsymbol{A}$ and the wave-number $k \equiv \omega/c$. The fields all vary with times as $\exp i\omega t$ but this factor will usually be omitted here.

It proves possible, and helpful, to treat the propagation of a paraxial beam which satisfies this equation by treating the beam as a superposition of "beam-modes" each of which retains, as it propagates, a characteristic pattern of distribution of field amplitude and phase in successive cross-sections, albeit with a changing scale [1]–[3]. The relative amplitudes of the beam-modes making up a beam can be determined if the field of the beam is known over some cross-section through the beam. We shall refer to this cross-section as the source-plane; it will usually be the aperture plane of a feed-horn (not only for a transmitting system but also for a receiving system because the feed-horn in which the detector is mounted will usually be the beam-defining component). We shall use phraseology appropriate to transmitting systems but time-reversal allows a direct translation for receiving systems.

We shall not deal with the explicit determination of the beam-mode amplitudes for any particular horn here; our objective is to establish a systematic basis for the analysis of the transformations suffered by individual beam-modes as they propagate through a system of (ideal) lenses or conic-section reflectors. For the design of high performance systems it is necessary to retain detailed knowledge, not only of the fundamental beam-mode, but also of higher-order beam-modes. The relative phases of the modes then become a crucial

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matter and we shall place some emphasis on phase in the development of beam-mode analysis in this and later Sections.

Our first purpose is to establish the beam-mode representation for a paraxial beam propagating into the half-space $z > 0$ from a source plane $z = 0$ and to do this in a way that allows the vector properties of the field to be recoverable. The original demonstration of beam-mode analysis [1] was fully vectorial but involved rather opaque formalism. Most subsequent demonstrations [2]–[6] have been made in the context of scalar field representations (but see [7]). To have a formulation that allows recovery of the vector properties of the fields we develop our treatment of beam-modes from a representation of a beam in terms of its angular spectrum of plane-waves. This also allows a clear view of the limitations imposed by the paraxial assumption.

2.2 Angular Spectrum of Plane-Waves

Consider a propagating beam-field whose form in the xy-plane at $z = 0$ is known. This known form will serve as the boundary condition necessary for finding an explicit solution to the wave-equation representing the beam-field propagating into the half-space, $z > 0$, which is assumed to be source-free.

A plane-wave is a particular solution to the wave-equation; a general solution can be written as a superposition of plane-waves travelling in all possible directions into the half-space $z > 0$, with two orthogonally polarized waves for each such direction [8]–[10]. We can choose, for the orthogonal polarizations for each direction, linear polarizations, one having zero y-component of the electric field and the other zero x-component. If we denote these two polarizations P and Q respectively we have the following two plane-wave superpositions to represent the electric field in the propagating beam-field (omitting the factor $\exp i\omega t$)

$$\begin{aligned} E_x(x,y,z) = & \int\int_{-\infty}^{+\infty} P(k_x,k_y) \\ & \cdot \exp - i(k_x x + k_y y + k_z z) \cdot dk_x\, dk_y \\ E_y(x,y,z) = & \int\int_{-\infty}^{+\infty} Q(k_x,k_y) \\ & \cdot \exp - i(k_x x + k_y y + k_z z) \cdot dk_x\, dk_y\,. \end{aligned} \tag{2}$$

Each is an integral over an angular spectrum of plane-waves, the angular spectra being defined by the direction dependent plane-wave amplitudes $P(k_x, k_y)$ and $Q(k_x, k_y)$. The k_x, k_y, k_z are components of the wave-vector, $\boldsymbol{k}$, with direction-independent magnitude $k = (k_x^2 + k_y^2 + k_z^2)^{1/2}$.

To determine the explicit forms of $P(k_x, k_y)$ and $Q(k_x, k_y)$ by invoking the boundary condition, i.e. the known form of the field in the source-plane $z = 0$, we note from (2) that

$$\begin{aligned} E_x(x_0,y_0) = & \int\int_{-\infty}^{+\infty} P(k_x,k_y) \\ & \cdot \exp - i(k_x x_0 + k_y y_0) \cdot dk_x\, dk_y \\ E_y(x_0,y_0) = & \int\int_{-\infty}^{+\infty} Q(k_x,k_y) \\ & \cdot \exp - i(k_x x_0 + k_y y_0) \cdot dk_x\, dk_y \end{aligned} \tag{3}$$

where x_0, y_0 are position co-ordinates in the plane $z = 0$. These are two-dimensional Fourier transform relations. The scalar angular spectra $P(k_x, k_y)$ and $Q(k_x, k_y)$ are thus determined as the (inverse) Fourier transforms of the source-plane fields, i.e.:

$$\begin{aligned} P(k_x,k_y) = & \frac{1}{4\pi^2}\int\int_{-\infty}^{+\infty} E_x(x_0,y_0) \\ & \cdot \exp i(k_x x_0 + k_y y_0) \cdot dx_0\, dy_0 \\ Q(k_x,k_y) = & \frac{1}{4\pi^2}\int\int_{-\infty}^{+\infty} E_y(x_0,y_0) \\ & \cdot \exp i(k_x x_0 + k_y y_0) \cdot dx_0\, dy_0\,. \end{aligned} \tag{4}$$

(The presence of the factor $1/4\pi^2$ indicates the Fourier transform convention we are using).

Once $E_x(x, y, z)$ and $E_y(x, y, z)$ are established in this way, reference to Maxwell's equations would serve to determine from them the remaining component of $\boldsymbol{E}$, $E_z(x, y, z)$, and the components of $\boldsymbol{H}$. Thus the two scalar angular spectra $P(k_x, k_y)$ and $Q(k_x, k_y)$ completely describe the field throughout the half-space $z > 0$.

It should be remarked that linearly polarized $\boldsymbol{E}$ fields are not the only possible choice as a plane-wave basis for description of the beam. For example, the two scalar fields representing left and right-hand circular polarizations could be used; and the $\boldsymbol{H}$-field, or the vector potential $\boldsymbol{A}$-field, could be taken instead of $\boldsymbol{E}$, since $\boldsymbol{H}$ and $\boldsymbol{A}$ as well as $\boldsymbol{E}$ obey the vector Helmholtz equation. One of the possible choices would eventually prove to be computationally more economical in any particular case than the others in that it required fewer beam-modes in superposition to fit the beam well. For the beams encountered in millimeter-wave systems the choice of linearly polarized $\boldsymbol{E}$-fields is usually the best choice.

The concept of the angular spectrum of plane-waves is useful, as we show below, as a lead-in to beam-modes. It has an important application in its own right, however. If the properties of a planar optical component (such as a filter) are understood in terms of the changes in amplitude and phase that it produces in a true plane-wave incident upon it, the effect of that component on a beam of finite width can be calculated by decomposing that beam into its angular spectrum and assessing the changes made by the component on each of the plane-wave constituents. That is, the beam samples the plane-wave properties of the component over a range of angles of incidence; given the Fourier transform relationship between field and angular spectrum it will be clear that the smaller the beam-width the larger the angular range sampled. The modification of the angular spectrum by a planar optical component would not depend on its longitudinal location in the beam since the one angular spectrum applies throughout the $z > 0$ space (there is, for example, no advantage in placing such a component in a cross-section where the phase-fronts of the beam are plane, apart from the fact that the diameter of the component corresponding to a given level of truncation of the beam might be smaller there).

Equation (4) above shows the relationship of a beam's angular spectra to its near-field, i.e. to its source-plane field; (5)–(7) below show the very direct relationship of the angular spectra to the far-field. The field amplitude at a distant point in a particular direction is essentially the amplitude of the plane-wave in the angular spectrum which has its **k**-vector in that direction since application of steepest descent or stationary phase methods to (2) with $k\rho >> 1$ gives [11]

$$E_x(\theta,\phi) = i2\pi k^2 \left\{ \frac{\exp - ik\rho}{k\rho} \right\} \cdot \cos\theta \cdot P(\theta,\phi) \quad (5)$$

$$E_y(\theta,\phi) = i2\pi k^2 \left\{ \frac{\exp - ik\rho}{k\rho} \right\} \cdot \cos\theta \cdot Q(\theta,\phi) . \quad (6)$$

Here ρ, θ, ϕ are spherical polar co-ordinates and $k_x = k\sin\theta\cos\phi$, $k_y = k\sin\theta\sin\phi$; $P(\theta,\phi)$, $Q(\theta,\phi)$ denote $P(k_x,k_y)$, $Q(k_x,k_y)$ with these substitutions. It follows that the polar components of $\boldsymbol{E}$ in the far-field, $k\rho >> 1$, are

$$\begin{aligned}[E_r, E_\theta, E_\phi] = i2\pi k^2 \left\{ \frac{\exp - ik\rho}{k\rho} \right\} \\ \cdot \big[0,\ \cos\phi \cdot P(\theta,\phi) + \sin\phi \cdot Q(\theta,\phi), \\ - \cos\theta\sin\phi \cdot P(\theta,\phi) \\ + \cos\theta\cos\phi \cdot Q(\theta,\phi)\big] . \end{aligned} \quad (7)$$

The factor $\{\exp -ik\rho/k\rho\}$ in the expressions above has the form of a spherical phase-front but the beam's far-field will only have truly spherical phase-fronts when $P(k_x,k_y)$ and $Q(k_x,k_y)$ are real or pure imaginary and that would be so only for a field in the source-plane that has uniform phase and an amplitude distribution with a center of inversion symmetry.

The angular-spectra of plane-waves representation gives a complete description of a beam-field; but it does not have a form that allows direct calculation of the modification of a beam produced by a conic-section reflector or a lens (and the same could be said of alternative representations in terms of superpositions of cylindrical-waves or of spherical-harmonics). The beam-mode representation, and modal transfer matrices, were developed to this end [1]–[7]. We present a particular formulation of Beam-mode Optics appropriate for applications in the design of long-wave optical systems in the following three Sections.

2.3 Beam-Modes

We now seek a modal representation of *each* of the two scalar beam-fields of (2). We denote the scalar field by $\psi(x,y,z)$ and it angular spectrum by $A(k_x,k_y)$. It proves to be helpful to represent the beam-field as a modulated plane-wave, i.e.

$$\psi(x,y,z) = u(x,y,z) \cdot \exp - ikz \quad (8)$$

and to concentrate on the function $u(x,y,z)$. The relationship between $u(x,y,z)$ and $A(k_x,k_y)$ can then be presented as a Fourier transform thus

$$\begin{aligned} u(x,y;z) = \int_{-\infty}^{+\infty} \{A(k_x,k_y)\exp i(k-k_z)z\} \\ \cdot \exp - i(k_x x + k_y y)\, dk_x\, dk_y \end{aligned} \quad (9)$$

i.e. the field in the plane z is evaluated as the two-dimensional inverse Fourier transform of $\{A(k_x,k_y)\exp i(k-k_z)z\}$, z being treated parametrically,

$$u(x,y;z) = \mathrm{FT}\{A(k_x,k_y)\exp i(k-k_z)z\} \quad (10)$$

where $k_z = \left(k^2 - k_x^2 - k_y^2\right)^{1/2}$ and is positive (and real) when $\left(k_x^2 + k_y^2\right) < k^2$ (the beam propagates *into* the space $z > 0$) and is negative (and imaginary) for $\left(k_x^2 + k_y^2\right) > k^2$ (giving evanescent waves which exponentially decrease rather than increase). This means, of course (noting that the inverse Fourier transform of $A(k_x,k_y)$ is the field in the plane $z = 0$) that $u(x,y;z)$ is the convolution of the field in the plane $z = 0$ with the inverse Fourier transform of $\exp i(k-k_z)z$,

$$u(x,y;z) = u(x,y;0) * \mathrm{FT}\{\exp i(k-k_z)z\} . \quad (11)$$

This equation is the essential relationship governing the propagation. However, it does not allow useful explicit evaluation without invoking a paraxial assumption concerning the beam's angular spectrum, namely that $A(k_x,k_y)$ falls sufficiently with increasing k_x, k_y for it to be safe, if evaluating $u(x,y;z)$ from (10) above, to neglect all but the first term in an expansion of the second factor, $\exp i(k-k_z)z$, in powers of $\left(k_x^2+k_y^2\right)/k^2$. That is

$$\exp i(k-k_z)z = \exp i \left\{ kz - k\left(1 - \frac{k_x^2+k_y^2}{k^2}\right)^{\frac{1}{2}} z \right\} \quad (12)$$

$$= \exp i \left\{ \frac{k_x^2+k_y^2}{2k} z + \cdots \right\} \quad (13)$$

and if we retain only the first term in the expansion (the "paraxial assumption", the consequences of which we examine in Section 2.4) we have

$$u(x,y;z) = u(x,y;0) * \mathrm{FT}\left\{ \exp i \frac{k_x^2+k_y^2}{2k} z \right\} \quad (14)$$

i.e.

$$u(x,y;z) = u(x,y;0) * \frac{i2\pi k}{z} \exp \frac{-ik(x^2+y^2)}{2z} \quad (15)$$

It is helpful now to remove the specific choice $z = 0$ for the source plane. The field in the plane z is related to that in a source plane at $z = z_S$, say, by

$$\begin{aligned} u(x,y;z) = u(x,y;z_S) \\ * \frac{i2\pi k}{z - z_S} \exp \frac{-ik(x^2+y^2)}{2(z-z_S)} \end{aligned} \quad (16)$$

There is advantage in representing the source field $u(x,y;z_S)$ as a superposition of Gauss–Hermite (GH) functions; this is because such functions form a complete set (so any well-behaved function can be represented by such a superposition, with appropriate coefficients) and because separately they have Fourier transform properties which are peculiarly adapted to the relationships between a beam-field and its angular spectrum, and convolution properties adapted to the propagation relationship above. (Gauss–Laguerre functions would be an alternative, adapted to cylindrical-wave rather than plane-wave representations but we shall not pursue that here). The normalized two-dimensional Gauss–Hermite functions are

$$u_{mn}(x,y;z_S) = \left(2^{m+n-1}\pi m!\,n!\right)^{-\frac{1}{2}} \frac{1}{w_{Sx}^{\frac{1}{2}} w_{Sy}^{\frac{1}{2}}} \cdot \left\{ H_m\left(\sqrt{2}\,\frac{x}{w_{Sx}}\right) \cdot \exp - \frac{x^2}{w_{Sx}^2} \right\} \cdot \left\{ \exp - i\frac{kx^2}{2R_{Sx}} \right\} \left\{ \exp i\left(m+\frac{1}{2}\right)\Theta_{Sx} \right\} \cdot \left\{ H_n\left(\sqrt{2}\,\frac{y}{w_{Sy}}\right) \cdot \exp - \frac{y^2}{w_{Sy}^2} \right\} \cdot \left\{ \exp - i\frac{ky^2}{2R_{Sy}} \right\} \left\{ \exp i\left(n+\frac{1}{2}\right)\Theta_{Sy} \right\} \tag{17}$$

where the $H_m(X)$ and $H_n(Y)$ denote Hermite polynomials of orders m, n respectively $(m, n = 0, 1, 2, \cdots)$ [12]. The w_{Sx}, R_{Sx}, Θ_{Sx} and w_{Sy}, R_{Sy}, Θ_{Sy} are arbitrary real constants independent of x, y, m and n.

The field in the $z = z_S$ plane can be written as a superposition of the GH functions

$$u(x,y;z_S) = \sum_{m,n} C_{mn} u_{mn}(x,y;z_S); \qquad C_{mn} = \int\int_{-\infty}^{+\infty} u_{mn}^*(x,y;z_S) \cdot u(x,y;z_S)\,dx\,dy. \tag{18}$$

For some special forms of $u(x,y;z_S)$ appropriate choices for the values of w_{Sx}, w_{Sy}, R_{Sx}, R_{Sy}, Θ_{Sx} and Θ_{Sy} would give real coefficients C_{mn}; generally the C_{mn} will be complex. The freedom to choose values for w_{Sx}, w_{Sy}, R_{Sx}, R_{Sy}, Θ_{Sx}, Θ_{Sy} can be important in that an optimum choice will keep to a minimum the number of GH functions necessary for a good fit.

In order to simplify the expressions we have to develop we shall exercise some of the arbitrary choice allowed in assigning values to the parameters w_{Sx}, w_{Sy}, R_{Sx}, R_{Sy} and Θ_{Sx}, Θ_{Sy} by taking $w_{Sx} = w_{Sy}$, $R_{Sx} = R_{Sy}$ and $\Theta_{Sx} = \Theta_{Sy}$, i.e. reducing the number of parameters from six to three (w_S, R_S, Θ_S). This will not restrict our analyses to non-astigmatic beams because an astigmatic beam can be well fitted by a superposition of non-astigmatic GH functions albeit with a larger number than would be necessary with astigmatic GH functions. Our analyses below could be straightforwardly generalized to include astigmatic GH functions but with considerable elaboration in the expressions required. The superposition in (18) will therefore be taken hereon to involve the non-astigmatic GH functions (see equation 19 below)

Having written the field in the source-plane at $z = z_S$ as a superposition of GH functions, we can use the convolution above (16) to find the field in an arbitrary down-beam constant-z plane. A GH function convolves with a Gaussian function of imaginary argument to produce a particularly simple result, namely a GH function of the same order, with a scaled argument and added spherical phase-front curvature (Appendix A), i.e. (16) leads to the following form for the propagating field $u(x,y,z)$, namely a superposition of *beam-modes*

$$u(x,y,z) = \sum_{mn} C_{mn} u_{mn}(x,y,z) \tag{20}$$

where the coefficients C_{mn} are those introduced above and the $u_{mn}(x,y,z)$ are the well-known *Gauss–Hermite beam-modes* (see equation 21 below)
in which $w(z)$, $R(z)$ and $\Theta(z)$ are

$$\begin{aligned} w^2 &= w_0^2 + \{2(z-z_0)/kw_0\}^2 \\ R &= (z-z_0) + \left\{kw_0^2/2\right\}^2 / (z-z_0) \\ \Theta &= \tan^{-1}\left\{\frac{kw^2}{2R}\right\} + \Theta_0 \\ &= \sin^{-1}\left(1 + \left\{\frac{kw^2}{2R}\right\}^{-2}\right)^{-1/2} + \Theta_0 \end{aligned} \tag{22}$$

in which the constants w_0, z_0 and Θ_0 are clearly determined by the values assigned to w_S, R_S and Θ_S thus

$$u_{mn}(x,y;z_S) = \left(2^{m+n-1}\pi m!\,n!\right)^{-\frac{1}{2}} \frac{1}{w_S} \left\{ H_m\left(\sqrt{2}\,\frac{x}{w_S}\right) \cdot H_n\left(\sqrt{2}\,\frac{y}{w_S}\right) \cdot \exp - \frac{x^2+y^2}{w_S^2} \right\} \cdot \left\{ \exp - ik\frac{x^2+y^2}{2R_S} \right\} \{\exp i(m+n+1)\Theta_S\}. \tag{19}$$

$$u_{mn}(x,y,z) = \left(2^{m+n-1}\pi m!\,n!\right)^{-\frac{1}{2}} \frac{1}{w} \left\{ H_m\left(\sqrt{2}\,\frac{x}{w}\right) \cdot H_n\left(\sqrt{2}\,\frac{y}{w}\right) \cdot \exp - \frac{x^2+y^2}{w^2} \right\} \cdot \left\{ \exp - \frac{ik(x^2+y^2)}{2R} \right\} \{\exp i(m+n+1)\Theta\} \tag{21}$$

$$\begin{aligned} w_0^2 &= w_S^2 \Big/ \left\{1 + \left(kw_S^2/2R_S\right)^2\right\} \\ z_0 &= z_S - R_S \Big/ \left\{1 + \left(2R_S/kw_S^2\right)^2\right\} \\ \Theta_0 &= \Theta_S - \tan^{-1}\left\{\frac{kw_S^2}{2R_S}\right\} \\ &= \Theta_S - \sin^{-1}\left(1 + \left\{\frac{kw_S^2}{2R_S}\right\}^{-2}\right)^{-1/2}. \end{aligned} \tag{23}$$

(It is necessary to give the two forms for Θ in (22) in order to determine uniquely the quadrant for Θ.)

The first term in curly-brackets in (21) shows the form of the variation of the *modulus* of the mnth beam-mode over a cross-sectional plane. This is a Gaussian function of $(x^2 + y^2)$ modulated by the polynomials. The *scale* of this variation changes with z through the z-dependence of the beam-width parameter w. This expanding scale of the mode's field distribution as it propagates is the *diffractive spreading* of the beam.

The second term in curly-brackets shows the variation of the *phase* of the beam-mode field over a cross-sectional plane, *relative* to the on-axis value. The form of this term indicates (in paraxial approximation) a spherical phase-front with radius-of-curvature, R. The value of R varies with propagation distance, but not linearly and this means that the location of the centre-of-curvature of the beam-modes' equi-phase surfaces varies with down-beam distance (a second aspect of the diffractive spreading of the beam).

The third term in curly-brackets gives the *on-axis phase.* The phase-angle $(m + n + 1)\Theta$ registers an on-axis phase slip of the mn^{th} mode relative to a plane-wave phase $(kz - \omega t)$ with down-beam distance. This phase slippage is the third consequence of diffractive spreading. Θ is thus the "beam-mode phase-difference", i.e. the common on-axis phase-*difference* between *successive* modes.

The factor in front of the first curly-bracket normalizes the power carried, i.e. the integral of $u^*mn\ umn$ over any constant-z cross-section of the beam-mode has unity value.

At $z = z_0$, w takes its minimum value w_0 and $R \to \infty$, i.e. the phase-front is plane there; this is known as the beam-waist of the mode. If R_S is negative, $z_0 > z_S$; the beam converges as it propagates from $z = z_S$ to $z = z_0$ and diverges beyond z_0. If R_S is positive, $z_0 < z_S$, and the beam diverges as it propagates from $z = z_S$, as from a "virtual" beam-waist at $z = z_0$.

2.4 Paraxiality

Equation (11) shows that the field in a down-beam cross-section $u(x, y, z)$, is related to that in the source plane $z = z_S$, by convolution with the Fourier transform of $\exp i(k - k_z)(z - z_s)$; in arriving at the beam-mode representation the paraxial assumption was made, namely that only the leading term in the expansion of $\exp i(k - k_z)(z - z_s)$ in powers of $(k_x^2 + k_y^2)/k^2$ should be retained. If the next term in the expansion were now restored, each beam-mode field would have to be corrected by convolving it with the Fourier transform of $\exp\left\{i\frac{(k_x^2+k_y^2)}{8k^4}[k(z - z_S)]\right\}$. The larger $k(z - z_S)$ the broader the peak in the Fourier transform, and consequently the greater the modification produced by the convolution. Such a correction could be made numerically but we can draw some conclusions about whether it would be necessary to do so as follows.

The required corrections are, as we have noted above, greater at larger z, and are therefore most marked for the far-field, i.e. for $\hat{z} >> 1$ where $\hat{z} = 2(z - z_S)/kw_0^2$. For the far-field, however, an exact solution is available and we can assess directly the magnitude of the non-paraxiality correction. The exact solution given in (5), with $P(\theta, \phi)$ taking the form appropriate to a beam having, in the source plane, a uniform phase and a Gaussian amplitude distribution with width parameter w_0, gives the following expression for the far-field intensity $S_F = |E(\theta, \phi)|^2$, as a function of off-axis angle, θ, relative to the on-axis value

$$\frac{S_F(\theta)}{S_F(0)} = \cos^2\theta \exp\left\{-2\sin^2\theta/(2/kw_0)^2\right\}. \tag{24}$$

(The Fourier transform of a Gaussian function with width parameter w_0 is itself a Gaussian function with width parameter $2/w_0$; the $\sin^2\theta$ derives from the substitution $k^2\sin^2\theta$ for $(k_x^2 + k_y^2)$).

To compare this exact form with that indicated by a beam-mode solution it is necessary to put $(z - z_S) \to \infty$ in the beam-mode expression of (21) and to convert the result into a distribution over a constant-ρ (spherical) surface rather than over a constant-z plane. To do this it may be noted that the on-axis intensity decreases as $(z - z_S)^{-2}$ in the far-field, and $(x^2 + y^2)/(z - z_S)^2 \equiv \tan^2\theta$. Hence

$$\frac{S_F(\theta)}{S_F(0)} = \cos^{-2}\theta \exp\left\{-2\tan^2\theta/(2/kw_0)^2\right\}. \tag{25}$$

The two forms, the exact and that uncorrected for non-paraxiality, given by (24) and (25), are shown in Fig. 1 for four selected values of kw_0, namely 3, 4, 6 and 10. For $kw_0 = 10$ the difference is extremely small; for $kw_0 = 3$ it is significant. The value $kw_0 = 6$ marks a transition between needing no and needing some correction of the beam-mode solution for non-paraxiality.

For higher-order modes the far-field correction is of similar form to that applied to the fundamental—it is essentially the $\cos\theta$ factor in (5). However, for a given w_0, the outermost peaks in GH functions occur at greater distances from the axis the larger the mode-numbers. In the angular spectrum, the outermost peaks are at $\theta \sim \left(\frac{n}{2}\right)^{1/2}\frac{2}{kw_0}$, where n is one of the mode-numbers. The $\cos\theta$ non-paraxiality correction factor therefore has more consequence for a higher-order mode than for a lower; non-paraxial corrections may not be negligible for $kw_0 < 6(n/2)^{1/2}$.

An optical system which is to contain planar components such as filters and diplexers would usually be designed with $kw_0 >> 6$, not simply to avoid paraxiality but because such components will not provide high performance over a wide range of plane-wave angle-of-incidence. The requirement on kw_0 set by this consideration would usually be more demanding than the paraxial condition $kw_0 > 6(n/2)^{1/2}$ which corresponds to an angular spread $\theta \lesssim 1/3$ radians. An

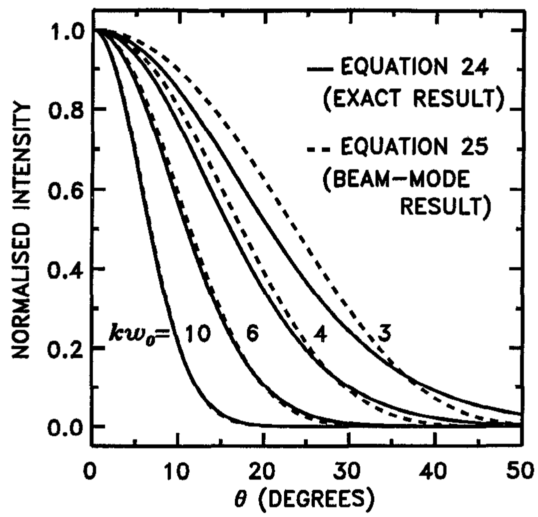

Fig. 1. Normalized far-field intensity distribution for a beam with a Gaussian distribution in the source plane. The parameter is kw_0.

exception to this might be at a transmit or receive antenna which, in order to have high directivity, would be electrically large and which would have to be illuminated by a strongly divergent beam to avoid having a system of appreciable length. In that case, however, the large antenna would usually be in the far-field of the beam that illuminates it and the non-paraxiality can be corrected as shown above.

In the literature there are several analyses of non-paraxiality (see for example [13], [14]) which keep the beam-mode in the far-field and modify the form in the near-field. The transmit and receive beams of real optical systems are formed by a near-field component, i.e. the feed-horn, however, and what is required is retention of the beam-mode form in the near-field with corrections applied in the far-field (see [15]).

2.5 Beam-mode Propagation Through Optical Systems

The function of a lens or of a conic-section reflector in an optical system can be idealized as the introduction of a phase-delay which varies quadratically with off-axis distance; such an "ideal phase-transformer" acts on each incident beam-mode by discretely changing the radius of curvature of its spherical phase-front from the incident value R_i to an emergent value R_e such that

$$\frac{1}{R_i} - \frac{1}{R_e} = \frac{1}{f} \tag{26}$$

where f is the "focal length" of the transformer, while leaving the amplitude distribution across the beam unchanged. (By convention, the phase-front radius of curvature is positive where the beam-mode is diverging and negative where it is converging. The focal length may be positive or negative). The focal length could be different for the xz and yz planes but we shall treat only axially-symmetric transformers.

Real lenses and conic-section reflectors will depart somewhat from this ideal behavior. A lens introduces some modification of the amplitude distribution as a result of reflection at its surfaces; a reflector gives some aberration of the emergent phase-front if used off-axis. A well-designed bloomed lens [16] can approach the ideal at least over a restricted range of frequency, however, and a well-designed reflector will give only weak aberration over a wide frequency range and the residual phase distortions can be cancelled to a considerable degree at other reflectors in the system if appropriately configured [17]. We shall assume ideal "phase-transformers" here.

Reflection at an off-axis conic-section reflector will change the directions of propagation and of polarization of the beam-modes, of course. If, as we assume hereon, the co-ordinate frame is rotated appropriately (see for example [5]) at each reflection in accord with metallic boundary conditions and the local plane-wave assumption (which presumes that reflector and phase-front radii of curvature are much smaller than $\lambda = 2\pi c/\omega$) the directions of propagation and polarization relative to the local co-ordinate frame will be unchanged.

The changes in the beam-mode parameters w, R, Θ that occur as the beam passes through an ideal transformer as indicated above can be simply stated:

$$w_e = w_i; \qquad \frac{1}{R_e} = \frac{1}{R_i} - \frac{1}{f}; \qquad \Theta_e = \Theta_i \tag{27}$$

where the subscripts i, e indicate the cross-sectional planes at the transformer, on the incidence and emergence sides respectively. The beam propagating away from the transformer is made up of beam-modes with unchanged coefficients, C_{mn}, with the emergence side of the transformer as a new source-plane so that the beam-mode parameters w, R, Θ vary with propagation distance from that plane in accordance with (22) but with new values for the constants w_0, R_0, Θ_0 given by (23) with w_S, R_S, Θ_S replaced by the w_e, R_e, Θ_e above. If this beam propagates to another ideal transformer, corresponding changes in w_0, R_0 and Θ_0 will occur there but, again, there will be no change in the C_{mn}.

The changes in the values of w, R, Θ as a beam propagates from one arbitrary cross-section to another, through a train of ideal phase-transformers, can be determined by a direct transfer matrix calculation if one introduces two particular combinations of w, R, Θ which show simple bilinear behavior both at an ideal transformer and in a free-space propagation step; they are

$$u \equiv w \exp - i\Theta \quad \text{and} \quad v \equiv \frac{w}{q} \exp - i\Theta \tag{28}$$

where q is given by

$$\frac{1}{q} = \frac{1}{R}\left(1 - i\left\{\frac{kw^2}{2R}\right\}^{-1}\right). \tag{29}$$

In terms of u and v, (27) for the changes in w, R, Θ at an ideal transformer becomes $u_e = u_i$ and $v_e = (-1/f)u_i + v_i$; (22) for free-space propagation becomes $u = u_0 + v(z - z_0)$ and $v = v_0$, where u_0, v_0 are the values of u, v at $z = z_0$, and hence $u_e = u_i + dv_i$ and $v_e = v_i$, where the subscripts i, e denote the beginning and end of a free-space step of length d. These bilinear relations can be expressed in matrix form:

At a transformer of focal length f:

$$\begin{bmatrix} u_e \\ v_e \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} u_i \\ v_i \end{bmatrix} \tag{30}$$

In a free propagation step of length d:

$$\begin{bmatrix} u_e \\ v_e \end{bmatrix} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_i \\ v_i \end{bmatrix} \tag{31}$$

If a beam is passing through a train of phase-transformers the output u_e, v_e from one step becomes the input u_i, v_i for the next. The overall transformation produced by the train is expressed in terms of a transfer matrix, known as the "$ABCD$ matrix" [18], [6] which is obtained by multiplying the matrices of the individual steps, free-space propagation steps alternating with phase transformations.

$$\begin{bmatrix} u_e \\ v_e \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} u_i \\ v_i \end{bmatrix} \tag{32}$$

It should be noted that the elements of the $ABCD$ matrices introduced here will all be real. It can also be useful to know that the determinant $(AD - CB)$ is equal to 1 for all of them [6].

Our purpose is to be able to determine the values of w, R, Θ in any specified output plane in terms of the values of w, R, Θ in any given input plane. Reference to the definitions of u, v and q in (28) and (29) leads directly to the following expressions for the ratios of output to input parameters (in deriving these relations it is noted that w is real positive).

$$\frac{w_e}{w_i} = \frac{|u_e|}{|u_i|} |A + B(1/q_i)| \tag{33}$$

$$\frac{R_e}{R_i} = \frac{\mathcal{R}(1/q_i)}{\mathcal{R}(1/q_e)} \quad \text{where } \frac{1}{q_e} = \frac{C + D(1/q_i)}{A + B(1/q_i)} \tag{34}$$

$$(\Theta_e - \Theta_i) = -\text{phase angle of } (A + B(1/q_i)) \tag{35}$$

$\mathcal{R}$ denotes the real part of the quantity in brackets.

If the matrix $A'B'C'D'$ denotes that calculated for a path beginning at the "input port" of a system and ending at the "output port" (these ports might coincide with the first and last transformers in the train but they need not), the elements $A'B'C'D'$ will depend only on the separations of the ports and transformers and on the transformers' focal lengths; $A'B'C'D'$ characterizes the train i.e. the optical instrument. If the matter of interest is then the beam-fields in arbitrarily chosen input and output planes, at distances d_i and d_e from the corresponding ports, the overall $ABCD$ matrix required is then

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & d_e \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} \begin{bmatrix} 1 & d_i \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} A' + C'd_e & A'd_i + D'd_e + C'd_i d_e + B' \\ C' & D' + C'd_i \end{bmatrix}. \tag{36}$$

The simplest example is a single phase-transformer with arbitrarily chosen input and output planes, distance d_i and d_e from the transformer respectively. The $ABCD$ matrix for this system is

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & d_e \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} 1 & d_i \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 - \left(\frac{d_e}{f}\right) & f - f\left(1 - \frac{d_i}{f}\right)\left(1 - \frac{d_e}{f}\right) \\ -\frac{1}{f} & 1 - \left(\frac{d_i}{f}\right) \end{bmatrix} \tag{37}$$

To illustrate the use of this transfer matrix consider an incident beam-mode which has its beam-waist in the input plane. We can determine the size and location of the beam-waist of the emergent beam-mode, in terms of the corresponding quantities for the incident beam-mode, as follows.

To find the location of the beam-waist of the emergent beam-mode, note that $(1/q)$ is pure imaginary at a beam-waist and use (34) for $1/q_e$ to determine the value of d_e which, given a pure imaginary $1/q_i$ at d_i, results in a pure imaginary $1/q_e$. The result is

$$(d_e - f) = \left[\frac{f^2}{(d_i - f)^2 + \left(\frac{kw_{0i}^2}{2}\right)^2} \right] (d_i - f). \tag{38}$$

Negative d_i or d_e would indicate a *virtual* beam-waist. Using this result, A and B in the matrix of (37) can be expressed in terms of f and $(d_i - f)$ only, so that (33) leads directly to

$$w_{0e}^2 = \left[\frac{f^2}{(d_i - f)^2 + \left(\frac{kw_{0i}^2}{2}\right)^2} \right] w_{0i}^2. \tag{39}$$

These relations are well known [19]. The change in the on-axis phase, from incident to emergent beam-waist, has received less attention but is of importance when treating beams made up of more than a single beam-mode. Equations (35) and (38) lead directly to

$$\tan(\Theta_e - \Theta_i) = \frac{-kw_{0i}^2/2}{d_i - f} = \frac{-kw_{0e}^2/2}{d_e - f} \tag{40}$$

and $(\Theta_e - \Theta_i)$ will be in first or third quadrant for $d_i < f$ and in second or fourth quadrant for $d_i > f$.

We have set out in this Section the way in which the values of w, R, Θ in any specified cross-section of a beam passing through a train of ideal phase-transformers are determined. Having found them the total field there is obtained by superposing beam-modes using the coefficients C_{mn} established when fitting the field in the source plane. Θ is the beam-mode on-axis phase *difference* of course, the on-axis phase of the mn mode being $(m + n + 1)\Theta$. Such superpositions are relatively undemanding computationally—up to, say, 100 modes could readily be included in calculations based on a personal computer.

III. Fourier Optics

We now use the beam-mode transfer matrices to show that there are special choices of the input and output planes of a system for which the beam transformation is a Fourier transformation. These special choices have previously been established by detailed consideration of diffraction integrals using essentially the same paraxial assumption as that required for beam-mode analysis [10] and the Fourier transform relationships involved have come to be known as Fourier Optics. In deriving them here through beam-mode analysis, they become incorporated in the more generally applicable Beam-mode Optics, and the route to recovering the vector properties of the beam-field is made clear. This may facilitate the explicit use of the Fourier Optics results in the design of long-wave systems—up to now they have found wide use in short wave—visible and infrared—optics but little use has been made of them in the millimetric and terahertz ranges. They have especial importance in the design of very wide-band long-wave systems. For those cases in which they are applicable, they circumvent the need to undertake a beam-mode decomposition explicitly.

Consider first a single phase-transformer and choose input and output planes which are both at the focal distance f from the transformer—the so-called front and back focal planes. The beam-mode transfer matrix for this case is (setting $d_i = d_e = f$ in the matrix of (37))

$$\begin{bmatrix} 0 & f \\ -\frac{1}{f} & 0 \end{bmatrix} \tag{41}$$

and this leads directly to $q_e q_i = -f^2$ and thence to the following relationships between the beam parameters w, R, Θ in the two focal planes

$$\begin{aligned} R_e R_i &= -f^2 \frac{1+\kappa^2}{\kappa^2}; \\ w_e^2 w_i^2 &= 4\left(\frac{f}{k}\right)^2 (1+\kappa^2); \\ \Theta_e - \Theta_i &= -\tan^{-1}\kappa + \frac{\pi}{2} \end{aligned} \tag{42}$$

where we have written κ for $kw_i^2/2R_i$. We have stressed earlier that the assignment of values to w, R in a reference or source-plane is arbitrary; choosing $R_i \to \infty$, $\kappa = 0$, makes these relationships especially simple: $w_e w_i = 2f/k$; $R_e \to \infty$ and $\Theta_e - \Theta_i = \pi/2$.

Relationships of this form have a special significance. It is shown in the Appendix that the Fourier transform of a Gauss-Hermite function is itself a Gauss-Hermite function of the same order number mn, whose parameters w, R, Θ are related to those of the initial G–H function precisely as indicated in (42) above when the conjugate variables of the function and its Fourier transform are identified as (x_i, y_i) and $((k/f)x_e, (k/f)y_e)$ respectively, where x_i, y_i are co-ordinates in the input plane and x_e, y_e co-ordinates in the output plane. This is true for a Gauss-Hermite mode of any mode number and, since Fourier transformation is a linear process, it will be true also of any superposition of beam-modes. That is to say, when an arbitrary paraxial beam passes through a phase-transformer, the field distribution in the back focal plane of the transformer is a Fourier transform of the field in the front focal plane, the conjugate variables being x_i, y_i and $(k/f)x_e, (k/f)y_e$.

The frequently invoked rule [19] that a beam-mode having a beam-waist in the front focal plane of a lens will produce an emergent beam having its beam-waist in the back focal plane, with $w_{0e} w_{0i} = 2f/k$ can be seen to be a simple special case of the more general and more powerful relation above.

The essential property of the transfer matrix above that leads to the Fourier transform relationship between the fields in the output and input planes is that the elements A and D are zero. If the elements of the matrix representing an arbitrary train of transformers, from input to output port, are A', B', C', D', then the planes at distances

$$d_e = -A'/C'; \qquad d_i = -D'/C' \tag{43}$$

from the input and output ports will give $A = 0$ and $D = 0$ in the system's $ABCD$ matrix (36). Such a choice therefore leads to the relationships in (42) above where f is now to be interpreted as

$$f = -1/C' = B'. \tag{44}$$

Thus the planes identified by (43) are the front and back focal planes of the train of phase-transformers, and an arbitrary paraxial beam propagating through the train will produce a field distribution in the back focal plane that is a Fourier transform of that in the front focal plane, the conjugate variable being (x_i, y_i) and $((k/f)x_e, (k/f)y_e)$.

Consider, now, two transformers separated by the sum of their focal lengths f_1, f_2 and take the input plane at a distance f_1 in front of the first transformer and the output plane at a distance f_2 beyond the second. The overall $ABCD$ matrix for this system is

$$\begin{bmatrix} 0 & f_2 \\ -\frac{1}{f_2} & 0 \end{bmatrix} \begin{bmatrix} 0 & f_1 \\ -\frac{1}{f_1} & 0 \end{bmatrix} = \begin{bmatrix} -\frac{f_2}{f_1} & 0 \\ 0 & -\frac{f_1}{f_2} \end{bmatrix}. \tag{45}$$

Equations 33–35 then lead to $q_e = (f_2/f_1)^2 q_i$ and thence the following relationships between the parameters of the fields in the output and input planes when a beam-mode passes through the system

$$w_e^2 = \left(\frac{f_2}{f_1}\right)^2 w_i^2; \qquad \frac{1}{R_e} = \left(\frac{f_1}{f_2}\right)^2 \frac{1}{R_i} \tag{46}$$

and

$$\begin{aligned} \Theta_e - \Theta_i &= \pi \text{ when } f_1, f_2 \text{ are of the same sign} \\ &= 0 \text{ when } f_1, f_2 \text{ are of different sign.} \end{aligned} \tag{47}$$

When f_1, f_2 are of different sign the system clearly produces in its output plane a coherent image of the field in its input plane. When f_1, f_2 are of the same sign the change in the phase difference between the mode mn and the fundamental mode is

$$(m+n)(\Theta_e - \Theta_i) = (m+n)\pi. \tag{48}$$

A mode for which $(m+n)$ is odd thus shows a π phase-change, i.e. an amplitude inversion, whereas a mode with $(m+n)$ even shows no change. This can simply be interpreted as inversion of the mode pattern, through the on-axis point, for all modes odd and even: the π phase-change for $(m+n)$ odd *is* inversion, and the zero phase-change for $(m+n)$ even is consistent with inversion because such a mode has inversion symmetry. The system similarly inverts and scales the field distribution of each mode.

This system thus produces in the output plane a magnified *coherent image* of the field in the input plane, i.e. phase as well as amplitude is reproduced, with scaling

$$x_i \rightarrow -\frac{f_1}{f_2}x_e; \qquad y_i \rightarrow -\frac{f_1}{f_2}y_e\,. \tag{49}$$

independent of k, i.e. the scaling is the *same for all frequencies.*

The well known fact that a beam-mode which has a beam-waist in the input plane of a two-lens system of this kind will give an emergent mode with its beam-waist in the output plane, with the output and input beam-waist parameters related by $w_{0e} = (f_2/f_1)w_{0i}$ independently of frequency [19] can now be seen to be a simple special case of the more general and powerful relationship demonstrated above.

The essential properties of the transfer matrix in (45) that lead to coherent imaging for this system are the zero values of the elements B and C. Any system having a transfer matrix with $B = C = 0$ will give coherent imaging.

We should refer to one more Fourier Optics theorem relating to a single phase-transformer, with an arbitrary input plane at distance d_i, and an output plane at the distance d_e such that

$$\frac{1}{d_e} = \frac{1}{f} - \frac{1}{d_i} \tag{50}$$

i.e. $B = 0$. (Negative values for d_e or d_i would imply "virtual" planes.) For this case the single-transformer transfer matrix (37) reduces to

$$\begin{bmatrix} -\frac{f}{\Delta} & 0 \\ -\frac{1}{f} & -\frac{\Delta}{f} \end{bmatrix} \tag{51}$$

where we have written Δ for $(d_i - f)$. Thence

$$\frac{1}{q_e} = \frac{\Delta}{f^2} + \frac{\Delta^2}{f^2}\frac{1}{q_i}$$

so that

$$w_e^2 = \left(\frac{f}{\Delta}\right)^2 w_i^2; \qquad \frac{1}{R_e} = \frac{\Delta}{f^2} + \left(\frac{\Delta}{f}\right)^2 \frac{1}{R_i} \tag{52}$$

and

$$\begin{aligned} \Theta_e - \Theta_i &= \pi \text{ when } \Delta \text{ and } f \text{ are of the same sign} \\ &= 0 \text{ when } \Delta \text{ and } f \text{ are of different sign.} \end{aligned} \tag{53}$$

These relationships are formally the same as those obtained with the two-transformer coherent-imaging system (46) and (47), with Δ, f replacing f_1, f_2, *apart from* the additional phase-front curvature term Δ/f^2. The arguments following (46) and (47) lead us to the conclusion that the field distribution in the output plane coherently images that in the input plane, with the substitutions

$$x_i \rightarrow -\frac{\Delta}{f}x_e; \qquad y_i \rightarrow -\frac{\Delta}{f}y_e \tag{54}$$

except that there is an additional spherical phase-factor

$$\exp - i\frac{\Delta \cdot k\left(x_e^2 + y_e^2\right)}{2f} \tag{55}$$

multiplying the field in the image plane.

In seeking the special planes for which a train of phase-transformers will produce coherent imaging, or Fourier transformation, in the ways examined above, it can be helpful to note that beam-mode transfer matrices have the same forms as the matrices that govern the ray-tracing constructions of geometrical optics. In the latter context the elements of the column vectors on which the matrices operate are the off-axis distance of a ray, and its slope [4]–[6]. Beam-mode Optics must contain simple ray-optics as a high-frequency limit but the identity of the matrices required in the two Optics is not a transparent matter [18]. Given the identity, however, the simple ray-tracing constructions can be used as *geometrical,* (not geometrical-*optical*) representations of the properties of beam-mode transfer matrices. That is the front and back focal planes of a train of phase-transformers (with respect to which the train will produce a Fourier transformation) can be found by a ray-tracing construction involving incident parallel rays and the focussing of the emergent rays in the back focal plane, together with a similar construction with parallel rays incident from the opposite direction to give the front focal plane. Also, a system will give coherent imaging if, using a ray-tracing construction, it can be seen to be divisible into two sub-systems with the back focal plane of the first coincident with the front focal plane of the second. And the condition in (50) which results in coherent imaging with an additional spherical phase factor (55) can be seen to be the imaging condition of ray tracing.

IV. Partially-Coherent Signal Beams

4.1 Coherent-mode Representation of Partially-Coherent Beams

The beam-mode analysis set out in the preceding Sections deals with coherent signal beams (whether derived from a coherent source or selected by coherent detection of signal power incident from an incoherent source). The treatment of the passage of a partially coherent signal beam through an optical system has in the past been a matter of tracing the propagation of second-order field correlations [20]. A new approach to the description of partial coherence has been advanced recently by Wolf [21]. We show below how this new description makes possible the use of the Optics of coherent beams in the analysis of the propagation of partially-coherent beams through optical systems, resulting in a conceptually much clearer basis of understanding.

Wolf [21] has shown how a partially-coherent beam can be spectrally decomposed, the component at any given frequency, ω, being represented by an ensemble of linear superpositions, $V(\boldsymbol{r}, \omega)$, of a set of spatially coherent elementary propagating beam-fields, $\psi_n(\boldsymbol{r}, \omega)$,

$$V(\boldsymbol{r}, \omega) = \sum_n a_n(\omega) \cdot \psi_n(\boldsymbol{r}, \omega) \cdot \exp i\omega t\,, \tag{56}$$

where $\boldsymbol{r}$ denotes the position vector for points in the beam. The complex coefficients, $a_n(\omega)$, are random numbers, fluctuating in amplitude and phase over the ensemble, or equivalently over time. In particular the second-order moments are

$$\langle a_n^*(\omega) \cdot a_m(\omega) \rangle = \lambda_n(\omega) \cdot \delta_{nm} \qquad (57)$$

where the brackets denote an average over the ensemble, or equivalently over time, $\lambda_n(\omega)$ is a constant, and δ_{nm} is the Kronecker δ. The forms of the orthonormal modes $\psi_n(\boldsymbol{r}, \omega)$, and the values of the constants $\lambda_n(\omega)$, depend on the second-order statistical correlations of the field. If $\psi_n(\boldsymbol{r}_S, \omega)$ denotes $\psi_n(\boldsymbol{r}, \omega)$ over a source-plane, S, i.e. a cross-section through the beam, Wolf shows that the $\psi_n(\boldsymbol{r}_S, \omega)$, and the $\lambda_n(\omega)$ are respectively the eigenfunctions and eigenvalues of the following equation

$$\int_S W(\boldsymbol{r}_{S1}, \boldsymbol{r}_{S2}, \omega) \cdot \psi_n(\boldsymbol{r}_{S1}, \omega) \cdot dS_1 = \lambda_n(\omega) \cdot \psi_n(\boldsymbol{r}_{S2}, \omega) \qquad (58)$$

where the integral is over the cross-section, S, and $W(\boldsymbol{r}_{S1}, \boldsymbol{r}_{S2}, \omega)$ is the cross-spectral density in S, i.e. the time-to-frequency Fourier transform of the two-point, two-time correlation function of the field.

The modes $\psi_n(\boldsymbol{r}, \omega)$ are spatially fully coherent. The statistical characteristics of the beam field are, in this representation, entirely to be found in the statistical properties of the random coefficients, $a_n(\omega)$. The cross-spectral density for any two points $\boldsymbol{r}_1$, $\boldsymbol{r}_2$ in the propagating beam then is given by

$$W(\boldsymbol{r}_1, \boldsymbol{r}_2, \omega) \equiv \langle V^*(\boldsymbol{r}_1, \omega) \cdot V(\boldsymbol{r}_2, \omega) \rangle \qquad (59)$$

$$= \sum_n \lambda_n(\omega) \cdot \psi_n^*(\boldsymbol{r}_1, \omega) \psi_n(\boldsymbol{r}_2, \omega). \qquad (60)$$

In particular, the intensity distribution in the beam, which is W for $\boldsymbol{r}_1 \equiv \boldsymbol{r}_2$, is

$$S(\boldsymbol{r}, \omega) \equiv W(\boldsymbol{r}, \boldsymbol{r}, \omega) = \sum_n \lambda_n(\omega) |\psi_n(\boldsymbol{r}, \omega)|^2. \qquad (61)$$

(Note that, when the intensity distribution is the matter of interest, all that needs to be known about the $a_n(\omega)$ are their second-order moments, i.e. the λ_n).

There is, in this coherent-mode representation of partially-coherent signal beams, opportunity to use coherent beam-mode analysis to treat the propagation of such a beam though an optical system. To explore this matter further we first consider a special case of partial coherence.

4.2 Beam-Mode Treatment of Beams from a Gaussian-Schell Source

The coherent-mode representation of a partially-coherent field in a source-plane has been explicitly demonstrated by Starikov and Wolf [22] for a special case. This is the so-called Gaussian-Schell model source in which the intensity distribution function and the correlation function over the source-plane are both Gaussian in form. Starikov and Wolf treat a one-dimensional Gaussian-Schell (GS) source but for our purposes we have straightforwardly extended their treatment to cover a two-dimensional axially symmetric GS source (Appendix C). For such a source in the $z = 0$ plane the cross-spectral density, at a given frequency, ω, is of the form

$$W(x_1, y_1, x_2, y_2) = [I(x_1, y_1)]^{\frac{1}{2}} [I(x_2, y_2)]^{\frac{1}{2}} \cdot \mu(x_1 - x_2, y_1 - y_2), \qquad (62)$$

where the spectral intensity distribution, $I(x, y)$ is

$$I(x, y) = \mathcal{A} \exp\left(\frac{-2[x^2 + y^2]}{w_{0I}^2} \right), \qquad (63)$$

and the degree of spatial coherence $\mu(x_1 - x_2, y_1 - y_2)$ is

$$\mu(x_1 - x_2, y_1 - y_2) = \exp\left(\frac{-2\left[(x_1 - x_2)^2 + (y_1 - y_2)^2\right]}{w_{0\mu}^2} \right), \qquad (64)$$

i.e. Gaussian functions with width parameters w_{0I} and $w_{0\mu}$ respectively. (We omit explicit recognition that $\mathcal{A}$, w_{0I}, $w_{0\mu}$ may depend on frequency, ω, to simplify the expressions). That is

$$W(x_1, y_1, x_2, y_2) = \mathcal{A} \exp\left\{ -\frac{x_1^2 + y_1^2 + x_2^2 + y_2^2}{w_{0I}^2} \right\} \cdot \exp\left\{ -2\frac{(x_1 - x_2)^2 + (y_1 - y_2)^2}{w_{0\mu}^2} \right\}. \qquad (65)$$

For this case the eigenfunctions of equation 58 above prove to be (Appendix C) the Gauss-Hermite functions ψ_{mn} (we make explicit here the fact that the previous subscript, n, would stand for an ordered pair of positive integers, m, n, when the source is, as here, two-dimensional)

$$\psi_{mn}(x, y) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{1}{w_0} \left(\frac{1}{2^{m+n} m!\, n!}\right)^{\frac{1}{2}} \cdot H_m\left(\sqrt{2}\frac{x}{w_0}\right) H_n\left(\sqrt{2}\frac{y}{w_0}\right) \cdot \exp -\left(\frac{x^2 + y^2}{w_0^2}\right), \qquad (66)$$

where the constant w_0 is

$$w_0 = w_{0I}^{\frac{1}{2}} \left(\frac{w_{0I}^2 w_{0\mu}^2}{w_{0\mu}^2 + 4w_{0I}^2} \right)^{\frac{1}{4}}, \qquad (67)$$

and the eigenvalues, λ_{mn}, relative to the fundamental λ_{00}, are

$$\frac{\lambda_{mn}}{\lambda_{00}} = \left(\frac{\beta^2}{2} + 1 + \beta\left[\left(\frac{\beta}{2}\right)^2 + 1 \right]^{\frac{1}{2}} \right)^{-(m+n)} \quad \text{where } \beta = \frac{w_{0\mu}}{w_{0I}}. \qquad (68)$$

Thus since the field over the Gaussian-Schell source-plane at $z = 0$ can be represented as a superposition of Gauss-Hermite functions with uncorrelated random amplitudes we have an immediate entry into a beam-mode treatment of the beam that propagates away from the source into the space $z > 0$. It will be a superposition of Gauss-Hermite beam-modes, with uncorrelated random amplitudes for which the second-order moment of the amplitudes will be proportional to the λ_{mn} given by (68). The modes' common beam-waist parameter is w_0, as given by (67) in terms of the width parameters of the intensity and correlation distributions of the source, w_{0I} and $w_{0\mu}$ respectively. The beam-waists are located in the source plane $z = 0$.

When the Gaussian width of the degree of spatial coherence is very much greater than the Gaussian width of the intensity distribution, i.e. $w_{0\mu} >> w_{0I}$, we are approaching the coherent limit, indicated by $\beta >> 1$. In this limit, (67) and (68) reduce to

$$w_0 \to w_{0I} \quad \text{and} \quad \frac{\lambda_{mn}}{\lambda_{00}} \approx \frac{1}{\beta^{2(m+n)}}. \tag{69}$$

Consequently, for all $m \neq 0$ and $n \neq 0$, $\lambda_{mn} << \lambda_{00}$, and, in the coherent limit, the source is well approximated by the lowest order mode alone.

The incoherent limit is approached when $w_{0\mu} << w_{0I}$. In this case $\beta << 1$ and

$$w_0 \to \sqrt{\frac{w_{0I} w_{0\mu}}{2}} \quad \text{and} \quad \frac{\lambda_{mn}}{\lambda_{00}} \approx 1 - (m+n)\beta. \tag{70}$$

All modes for which $(m+n) \lesssim 1/\beta$ are needed to describe the source well. This implies that the number of modes needed is

$$N \approx \frac{1}{2\beta^2}. \tag{71}$$

The cross-spectral density over the constant-z plane at down-beam distance z from the source-plane is given by (59)

$$W(x_1, y_1, x_2, y_2; z) = \sum_{m,n} \lambda_{mn} \psi_{mn}^*(x_1, y_1; z) \psi_{mn}(x_2, y_2; z) \tag{72}$$

where the ψ_{mn} are the Gauss-Hermite beam-modes with the on-axis phase omitted since the random character of the mode amplitudes removes coherent interference between the modes. The modes keep their Gauss-Hermite forms as they propagate, with an increasing width parameter $w(z)$, and develop spherical phase-fronts of changing radius of curvature $R(z)$ (22); since the values of $w(z)$ and $R(z)$ are common to all the modes, the mode superposition in the cross-spectral density above retains the Gauss-Schell form of (65), with the addition of a spherical phase-term, i.e.

$$\begin{aligned} W(x_1, y_1, x_2, y_2; z) = \mathcal{A} \exp&\left\{ -\frac{x_1^2 + y_1^2 + x_2^2 + y_2^2}{w_I^2} \right\} \\ \cdot \exp&\left\{ -2 \frac{(x_1 - x_2)^2 + (y_1 - y_2)^2}{w_\mu^2} \right\} \\ \cdot \exp&\left\{ \frac{ik(x_1^2 + y_1^2 - x_2^2 - y_2^2)}{2R} \right\} \end{aligned} \tag{73}$$

The width parameters $w_I(z)$ and $w_\mu(z)$ will both scale as $w(z)$, so (22):

$$\begin{aligned} w_I^2(z) &= \left(\frac{w_{0I}^2}{w_0^2} \right) w^2(z) = w_{0I}^2 (1 + \hat{z}^2); \\ w_\mu^2(z) &= \left(\frac{w_{0\mu}^2}{w_0^2} \right) w^2(z) = w_{0\mu}^2 (1 + \hat{z}^2) \\ R(z) &= z(1 + \hat{z}^{-2}) \end{aligned} \tag{74}$$

where $\hat{z} \equiv 2z/kw_0^2$, w_0 being dependent on w_{0I} and on $w_{0\mu}$ as given by (67).

The intensity distribution in a constant-z cross-section is given by $W(x_1, y_1, x_2, y_2; z)$ with $x_1 = x_2$ and $y_1 = y_2$, and thus from (74) has an axially-symmetric Gaussian form at all down-beam distances z:

$$S(x, y; z) = \mathcal{A} \exp\left\{ -2 \frac{x^2 + y^2}{w_I^2} \right\}. \tag{75}$$

The z-dependence of the intensity width parameter, w_I, is given by (from (67) and (74))

$$w_I^2 = w_{0I}^2 + \frac{4z^2}{k^2} \left(\frac{1}{w_{0I}^2} + \frac{4}{w_{0\mu}^2} \right). \tag{76}$$

In the near-field ($z \to 0$) the width of the beam is clearly essentially w_{0I} regardless of the degree of coherence. In the far-field ($z \to \infty$) the angular spread of the beam, $\theta_F \equiv w_I/z$, is

$$\theta_F \to \frac{2}{k} \left(\frac{1}{w_{0I}^2} + \frac{4}{w_{0\mu}^2} \right)^{\frac{1}{2}}. \tag{77}$$

The extremes of high and low degree of coherence (i.e. $\beta >> 1$ and $\beta << 1$, respectively) are

$$\begin{aligned} \beta >> 1: \quad \theta_F &\to \frac{2}{kw_{0I}} \\ \beta << 1: \quad \theta_F &\to \frac{4}{kw_{0\mu}} \end{aligned} \tag{78}$$

That is, the angular spread for low coherence is determined by the Gaussian width parameter of the correlation function, and for high coherence by that of the intensity distribution. The near-to-far field transition distance, z_t, is

$$\begin{aligned} \beta >> 1: \quad z_t &= \frac{kw_{0I}^2}{2} \\ \beta << 1: \quad z_t &= \frac{\beta}{2} \frac{kw_{0I}^2}{2}. \end{aligned} \tag{79}$$

It is well-known that the form of the near-field and that of the far-field of a *coherent* source are related as follows

$$A\Omega = 4\pi^2/k^2 = \lambda^2 \text{ (coherent beam)} \tag{80}$$

where the solid-angle, Ω, is a measure of the angular divergence of the far-field power pattern and the area A a measure of the extent of the near-field beam. Specifically, $\Omega = P/B$ where P is the total power in the beam and B is the on-axis power per unit solid-angle in the far-field, and $A = P/I$ where I is the on-axis power per unit area in the near-field. For the partially coherent beam from a Gaussian-Schell source, the A and Ω defined in these ways are

$$A = \pi w_{0I}^2 \tag{81}$$

$$\Omega = \pi\theta_F^2 = \frac{4\pi}{(kw_{0I})^2}\left(1 + \frac{4}{\beta^2}\right). \tag{82}$$

Hence

$$A\Omega = \lambda^2\left(1 + \frac{4}{\beta^2}\right). \tag{83}$$

We have noted (71) that the number, N, of beam-modes required to describe well a partially coherent beam is $\sim 1/2\beta^2$; it is clear that this number is $\sim A\Omega/\lambda^2$.

The propagation of the partially coherent beam from a Gaussian-Schell source through a system of lenses or conic-section reflectors can now be treated in terms of the independent propagation of the elementary beam-modes as set out in Section 2.5. The values of w_{0I}, $w_{0\mu}$ which characterize the source determine, through (67), the value of w_0, the beam-waist parameter common to all the beam-modes. The $ABCD$ matrices will then allow the modal width parameter, w, and the phase-front curvature, R, both of which are common to all the beam-modes, to be determined for any selected cross-section through the beam. The modes' on-axis phases are not all the same but are of no concern here because the modes' complex amplitudes are fluctuating statistically and leave no correlations of on-axis phases. Having determined w for a selected cross-section, the values of w_I and w_μ then follow from (74). Knowing w_I, w_μ, R for a selected cross-section will allow the cross-spectral density in that cross-section to be determined (73). In particular, the intensity distribution in the cross-section, i.e. $W(x_1, y_1, x_2, y_2; z)$ for $x_1 = x_2$, $y_1 = y_2$, is as given in (75).

A system which includes beam-dividers and beam-combiners (an interferometer would be an example) may provide more than one path. In that case it would be necessary to use beam-mode analysis first to determine the field in the cross-section contributed by a single beam-mode from the source, adding coherently the fields of the beams arriving via the several paths. The resultant fields in the cross-section would then be used as the elementary coherent fields in the evaluation of the cross-spectral density adding the elementary fields incoherently with the relative weights λ_{mn}.

4.3 Beam-Mode and Fourier Optics for Partially-Coherent Beams

In the preceding Section we treated a special case of partial coherence for which the elementary coherent beam-fields proved to be Gauss-Hermite beam-modes. For beams from sources having other states of coherence the required elementary beam-fields will not be Gauss-Hermite beam-modes; nevertheless, beam-mode analysis can be used, at least in principle, because, as we have established earlier, any paraxial coherent beam can be represented as a superposition of Gauss-Hermite beam-modes. Provided, therefore, one can determine the elementary spatially-coherent beam-fields required to describe the beam from a given partially coherent source, each could be decomposed into constituent Gauss-Hermite beam-modes. The field in any selected cross-section through the beam in the optical system could then be determined by adding coherently the fields of the several Gauss-Hermite beam-modes that together make up each elementary beam-field of the source and then determining the cross-spectral density over the selected cross-section by adding the resultant fields incoherently. This may appear to be a large computational process since many elementary fields might be required. However, the powerful transformations relating the coherent fields in special planes of optical systems which are collectively known as Fourier Optics (Section 3) would usually be applicable and they remove the need for explicit analysis into beam-modes.

To make use of the Wolf representation of the beam from a partially-coherent secondary source it is necessary to know the statistical properties of the source (or of the detector), i.e. its cross-spectral density $W(\boldsymbol{r}_1, \boldsymbol{r}_2, \omega)$ or equivalently the elementary functions $\phi_{mn}(\boldsymbol{r}, \omega)$ and the eigenvalues λ_{mn}. That information may not be readily available for real sources.

If a hot black cavity is coupled to a (cool) wave-guide which then tapers into a horn the waveguide will transmit, at each frequency, in waveguide modes, each of which propagates into the tapered-horn without generating higher order modes, so that, in the aperture plane of the horn, each waveguide mode contributes a coherent field having a well-known amplitude distribution and a spherical phase-front centered at the apex of the horn. The thermal field in the hot cavity would excite each waveguide mode with fluctuating amplitude with statistical mean spectral power density equal to kT, where k is the Boltzmann constant and T the temperature of the cavity. The forms of the coherent fields over the aperture of the horn and the statistical means of the amplitudes are thus known and can respectively be identified with the elementary coherent fields, $\psi_{mn}(\boldsymbol{r}_S, \omega)$ over the source plane and the associated eigenvalues λ_{mn}. Murphy and Padman have considered the far-field antenna patterns of such sources [23] and their results can be interpreted along the lines traced here.

The surface of a hot body is a partially-coherent source. The pertinent correlation length of the fluctuations in the field at the surface would be $\sim \lambda = 2\pi/k$ corresponding to radiation over the full 2π solid angle. That is to say, the radiated field at a distance and over a solid angle approaching 2π would, if propagated back to the source in a time-reversed sense, reconstitute a field at the source which had a correlation length $\sim \lambda$; this might fail to reproduce all details of the field at the source because it would not include components of the field there which have spatial frequencies greater than k and which therefore generate evanescent fields (Section 2.3). If baffles with apertures were erected in front of an extended hot-body source so as to give a beam diverging over less than 2π solid angle, the effective correlation length in the source would be correspondingly greater. It would be possible to estimate the partial coherence properties of such a source in this way. Similar remarks would apply to the beam-determining properties of incoherent detectors.

V. Conclusion

We have presented a general analytical framework for the design of high-performance long-wave optical systems. This framework is the basis of methods currently being used in the design of systems for earth-remote-sensing, plasma diagnostics, atmospheric studies and astronomy (see for example [22]–[33]). We have said little about the effects of truncation of beams at apertures in the system, or about the aberrations and cross-polar contamination arising at off-axis reflectors. The former is usually a matter of determining the necessary diameters of apertures, lenses and reflectors to avoid significant beam contamination due to edge diffraction; and the latter a matter of configuring successive reflectors to compensate for the aberrations [17], [36], [16]. Both issues merit separate attention later.

The effective use of the design methodology set out here depends on being able to characterize the sources and detectors—i.e. the feed-horns or other beam-forming components—so that the relative amplitudes of the beam-modes can be determined. For a coherent system the field over the aperture of the horn is what is required (see for example [25], [37]). For incoherent sources or detectors the cross-spectral density would be necessary for precise analysis but the value of $A\Omega$ (83) is frequently all that is known.

Appendices

A. Convolution

We show below that the convolution of a Gauss-Hermite function

$$H_n(X)\cdot\exp\left(\frac{-X^2}{2}\right)\cdot\exp\left(\frac{-iBX^2}{2}\right) \tag{84}$$

with the Gaussian function

$$\exp\left(\frac{-iAX^2}{2}\right) \tag{85}$$

gives (for real A and B) a similar Gauss-Hermite function having a scaled argument and added curvature and phase, namely

$$\left(-\frac{2\pi i}{|\alpha|}\right)^{\frac{1}{2}}\cdot H_n(X')\cdot\exp\left(\frac{-X'^2}{2}\right)\cdot\exp\left(\frac{-iB'X'^2}{2}\right)\cdot\exp i\left(n+\frac{1}{2}\right)\theta \tag{86}$$

where

$$\begin{aligned}\alpha &\equiv 1+i(A+B)\\ B' &\equiv \{1+B(A+B)\}/A\\ X' &\equiv AX/|\alpha|\\ \theta &\equiv \tan^{-1}(A+B)^{-1}\\ &\equiv \sin^{-1}\left\{1+(A+B)^2\right\}^{\frac{1}{2}}.\end{aligned} \tag{87}$$

The convolution is the integral

$$\int_{-\infty}^{+\infty} H_n(X_0)\cdot\exp\left(\frac{-X_0^2}{2}\right)\cdot\exp\left(\frac{-iBX_0^2}{2}\right)\cdot\exp\left(\frac{-iA(X-X_0)^2}{2}\right)\cdot dX_0. \tag{88}$$

Following an extension of the method given in [7] we make use of the generating function for Hermite polynomials [12]

$$\sum_{n=0}^{\infty}\frac{s^n}{n!}H_n(X)=\exp(-s^2+2sX) \tag{89}$$

and first examine the integral

$$\begin{aligned}&\int_{-\infty}^{+\infty}\sum_{n=0}^{\infty}\frac{s^n}{n!}H_n(X_0)\cdot\exp\left(\frac{-X_0^2}{2}\right)\cdot\exp\left(\frac{-iBX_0^2}{2}\right)\cdot\exp\left(\frac{-iA(X-X_0)^2}{2}\right)\cdot dX_0\\ &=\int_{-\infty}^{+\infty}\exp(-s^2+2sX_0)\cdot\exp\left(\frac{-X_0^2}{2}\right)\cdot\exp\left(\frac{-iBX_0^2}{2}\right)\cdot\exp\left(\frac{-iA(X-X_0)^2}{2}\right)\cdot dX_0.\end{aligned} \tag{90}$$

which can be re-arranged, by taking outside the integral those terms independent of X_0 and completing a square inside the integral, to give

$$=\exp\left(-s'^2+2s'X'-\frac{X'^2}{2}-i\frac{B'X'^2}{2}\right)\cdot\int_{-\infty}^{+\infty}\exp-\left\{\left(\frac{\alpha}{2}\right)^{\frac{1}{2}}X_0-\frac{s+iAX/2}{(\alpha/2)^{\frac{1}{2}}}\right\}^2 dX_0 \tag{91}$$

where α, X' and B' are defined in (87) above and $s'\equiv s(1-\frac{2}{\alpha})^{1/2}$. The integral above can be directly evaluated to give $(2\pi/\alpha)^{1/2}$ and the first two terms in the exponential outside the integral can be recognized as a generating function for Hermite polynomials with argument X' so we continue with

$$=\left(\frac{2\pi}{\alpha}\right)^{\frac{1}{2}}\sum_{n=0}^{\infty}\frac{s^n(1-\frac{2}{\alpha})^{\frac{n}{2}}}{n!}\cdot H_n(X')\cdot\exp\left(\frac{-X'^2}{2}\right)\cdot\exp\left(\frac{-iB'X'^2}{2}\right). \tag{92}$$

Comparing corresponding terms in the summations in equations 90 and 92 we find the convolution integral reduces to

$$\left(\frac{2\pi}{\alpha}\right)^{\frac{1}{2}}\left(1-\frac{2}{\alpha}\right)^{\frac{n}{2}}\cdot H_n(X')\cdot\exp\left(\frac{-X'^2}{2}\right)\cdot\exp\left(\frac{-iB'X'^2}{2}\right). \tag{93}$$

The first two factors above together can be reduced to

$$\left(-\frac{2\pi i}{|\alpha|}\right)^{\frac{1}{2}}\cdot\exp i\left(n+\frac{1}{2}\right)\theta \tag{94}$$

where θ is defined in (87) above. This gives us the form for the convolution in (86) above.

If we now identify the Gauss-Hermite function in (84) with a GH component of the field of a beam in a reference plane $z = z_s$ (i.e., (19) and identify the function in (85) as $\exp\left(-ik_x^2/2(z - z_S)\right)$, we have the following connections

$$X^2 = \frac{2}{w_S^2}x^2; \qquad B = \frac{kw_S^2}{2R_S}; \qquad A = \frac{kw_S^2}{2(z - z_S)}. \tag{95}$$

Recalling the pre-factor $i2\pi k/(z - z_S)$ in the Fourier transform of $\exp\left(ik_x^2(z - z_S)/2k\right)$, and also the need to take the product of terms of the above form, one for y and the other for x, brings us to the expression for a beam-mode field given in (21).

B. Fourier Transformation

The Fourier transform of a Gauss-Hermite function

$$H_n(X) \cdot \exp\left(\frac{-X^2}{2}\right) \cdot \exp\left(\frac{-i\kappa X^2}{2}\right) \tag{96}$$

can be established by the same procedure as followed above for the convolution. The result is (where K is written for the conjugate variable to X)

$$\frac{1}{(2\pi)^{\frac{1}{2}}\eta^{\frac{1}{2}}} H_n\left(\frac{K}{\eta}\right) \cdot \exp\left(-\frac{K^2}{2\eta^2}\right) \cdot \exp\left(i\frac{\kappa K^2}{2\eta^2}\right) \cdot \exp - i\left\{\left(n + \frac{1}{2}\right)\phi - n\frac{\pi}{2}\right\} \tag{97}$$

where $\eta = \left(1 + \kappa^2\right)^{\frac{1}{2}}$ and $\phi = \tan^{-1}\kappa$. We see, in particular (putting $\kappa = 0$) that $H_n(X) \cdot \exp\left(-X^2/2\right)$ Fourier transforms to $1/(2\pi)^{\frac{1}{2}} \cdot H_n(K) \cdot \exp - \left(K^2/2\right) \cdot \exp(in\pi/2)$.

In order to bring this form into correspondence with the notation in Section III the following connections must be made

$$\frac{X^2}{2} \to \frac{x_i^2}{w_i^2}; \qquad \frac{\kappa X^2}{2} \to \frac{kx_i^2}{2R_i}; \qquad KX \to \left(\frac{k}{f}x_e\right)(x_i)$$
$$\frac{K^2}{2\eta^2} \to \frac{x_e^2}{w_e^2}; \qquad \frac{\kappa K^2}{2\eta^2} = -\frac{kx_e^2}{2R_e}. \tag{98}$$

These relations lead directly to equation 42 of Section 3.

C. Gaussian-Schell Source Elementary Functions

Starikov and Wolf [22] use the bilinear generating function for Hermite polynomials [38] to show that one of the orthogonal one-dimensional factors of the cross-spectral density given in (65), i.e.

$$W(x_1, x_2) = \mathcal{A}\exp\left\{-(a + b)\left(x_1^2 + x_2^2\right) + 2bx_1x_2\right\} \tag{99}$$

where $a \equiv 1/w_{0i}^2$ and $b \equiv 2/w_{0\mu}^2$, can be put into the form

$$W(x_1, x_2) = \sum_{m=0}^{\infty} \frac{\mathcal{A}}{2^m m!} \cdot \left(\frac{2c}{a + b + c}\right)\left(\frac{b}{a + b + c}\right)^m \cdot H_m\left(x_1\sqrt{2c}\right)H_m\left(x_2\sqrt{2c}\right) \cdot \exp\left[-c^2\left(x_1^2 + x_2^2\right)\right] \tag{100}$$

where $c = \left(a^2 + 2ab\right)^{1/2}$.

An expression for the other orthogonal one-dimensional factor of the cross-spectral density, $W(y_1, y_2)$, can be similarly obtained; the product of the two one-dimensional factors gives the two-dimensional cross-spectral density of the Gaussian-Schell source (equation 65) in the form

$$W(x_1, x_2, y_1, y_2) = \sum_{m,\, n=0}^{\infty} \frac{\mathcal{A}}{2^{m+n} m!\, n!} \cdot \left(\frac{2c}{a + b + c}\right)\left(\frac{b}{a + b + c}\right)^{m+n} \cdot H_m\left(x_1\sqrt{2c}\right)H_m\left(x_2\sqrt{2c}\right) \cdot H_n\left(y_1\sqrt{2c}\right)H_n\left(y_2\sqrt{2c}\right) \cdot \exp\left[-c^2\left(x_1^2 + x_2^2 + y_1^2 + y_2^2\right)\right]. \tag{101}$$

This is seen to be of the form of (60) for $z = 0$ (for a two-dimensional source the subscript n in (60) represents the ordered pair of positive integers, mn) if the eigenfunctions are identified as the Gauss-Hermite functions

$$\psi_{mn}(x, y) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} c^{\frac{1}{2}} \left(\frac{1}{2^{m+n} m!\, n!}\right)^{\frac{1}{2}} \cdot H_m\left(x\sqrt{2c}\right)H_n\left(y\sqrt{2c}\right)\exp - c\left(x^2 + y^2\right), \tag{102}$$

and the eigenvalues as

$$\lambda_{mn} = \mathcal{A}\left(\frac{\pi}{a + b + c}\right)\left(\frac{b}{a + b + c}\right)^{m+n}. \tag{103}$$

Equations (66) and (67) now follow from (102) when it is recalled that $a = 1/w_{0i}^2$, $b = 2/w_{0\mu}^2$ and c is identified as $1/w_0^2$; and (68) follows from (103).

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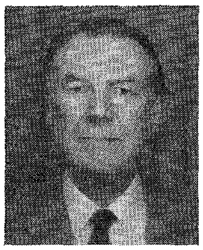

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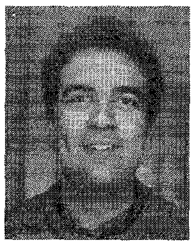

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